Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

```
Clear["Global`*"]
```

1 - 7 Contour integration. Unit Circle
Integrate counterclockwise around the unit circle.

1. $\oint_C \frac{\text{Sin}[z]}{z^4} \, dz$

```
Clear["Global`*"]
```

This problem is in the s.m., and I will use it. The explanation in the s.m. clarifies the procedure. Although I am looking at an integral symbol, there is really no integrating to be done in the normal sense of the term. On numbered line (1) of p. 664 of the text is the formula I will be working with, namely

$$f^{(n)}(z_0) == \frac{n!}{2\pi i} \oint_C \frac{f[z]}{(z-z_0)^{n+1}} \, dz$$

and rearranging and filling in parts of this equation will result in the answer to the problem. First step is to get the ‘integral’ term by itself

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f[z]}{(z-z_0)^{n+1}} \, dz$$

At this point, looking back and forth between the numbered line (1) ‘integral’ and the problem ‘integral’, I can identify components. Thus

```
f[z_] = Sin[z]
Sin[z]
```

Likewise, $z^4 == (z - z_0)^{n+1}$,
implying that $z_0 == 0$ and $n == 3$. So now I can identify

```
z0 = 0
0
```

```
f'''[z]
-Cos[z]
```

and

```
f'''[z0]
-1
```

So now all I need to do is put the lhs together

$$\frac{2\pi i}{3!} f'''[z_0]$$

$$-\frac{i\pi}{3}$$

The green cell matches the text answer. This is not difficult. I just need to remember that I am seeking the lhs, not the right. One thing I need to remember is that f[z] needs to be analytic for this procedure to work. No problem so far with that.

3. $\oint_C \frac{\text{Exp}[z]}{z^n} dz$, n = 1, 2, ...

```
Clear["Global`*"]
```

Same drill. Putting the working form of the numbered line (1) equation in here for reference:

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f[z]}{(z-z_0)^{n+1}} dz$$

```
f[z_] = Exp[z]
```

e^z

And $z^n == (z-z_0)^{n+1}$, implying that z_0==0 and n==n+1. What? How will I handle this? Pretty clear that I will have to do it one n-value at a time. So suppose I take n (in the problem integral) to be 3. Then n (in the template integral) will be 2. So I can write

z_0 = 0

0

f''[z]

e^z

and

f''[z_0]

1

Putting the lhs together,

$$\frac{2\pi i}{2!} f''[z_0]$$

$i\pi$

The text answer is the general sol'n, but yields the green cell when n=3. I could write the general sol'n too, $\left(\frac{2\pi i}{(n-1)!}\right) e^0$.

```
hf = HoldForm[Series[2 ⅈ π/(-1 + n)!, {n, 0, 4}]]
```

$$\text{Series}\left[\frac{2\,i\,\pi}{(-1+n)!},\ \{n,0,4\}\right]$$

Except this doesn't really give me what I want. I don't know how to make a series that looks like what I want.

5. $\oint_C \frac{\text{Cosh}[2\,z]}{\left(z-\frac{1}{2}\right)^4}\,dz$

```
Clear["Global`*"]
```

Pasting in the target expression again

$$\frac{2\,\pi\,i}{n!}\,f^{(n)}(z_0) = \oint_C \frac{f[z]}{(z-z_0)^{n+1}}\,dz$$

so that

```
f[z_] = Cosh[2 z]
```

Cosh[2 z]

and $\left(z-\frac{1}{2}\right)^4 == (z-z_0)^{n+1}$ implies that $z_0 = \frac{1}{2}$ and n==3

```
z0 = 1/2
```

$\frac{1}{2}$

$$\frac{2\,\pi\,i}{3!}\,f'''[z_0]$$

$$\frac{8}{3}\,i\,\pi\,\text{Sinh}[1]$$

```
N[%]
```

0. + 9.84534 ⅈ

The green cells above match the text answer.

7. $\oint_C \frac{\text{Cos}[z]}{z^{2n+1}}\,dz$, n = 0, 1, ...

```
Clear["Global`*"]
```

The target expression in this case is confusing, because the *n* on the left is not the *n* on the right

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f[z]}{(z-z_0)^{n+1}}\, dz$$

so that

```
f[z_] = Cos[z]
Cos[z]
```

and $z^{2n+1} = (z - z_0)^{m+1}$, which implies that $z_0 = 0$ and

```
Solve[2 n + 1 == m + 1, m]
{{m → 2 n}}
```

The way this is set up, both m and n need to be integers. To restate the target

$$\frac{2\pi i}{(2n)!} f^{(2n)}(z_0) = \oint_C \frac{f[z]}{(z-z_0)^{m+1}}\, dz$$

Here m can only be even, but calculations are not done on the right, but rather the left.

```
z0 = 0
0

f[z0]
1
```

$$\frac{2\pi i}{0!} f[z_0] \text{ /. } f[z_0] \to 1 \text{ (* for n=0 *)}$$

$2 i \pi$

$$\frac{2\pi i}{(2\times 1)!} f''[z_0] \text{ (* for n=1 *)}$$

$-i \pi$

$$\frac{2\pi i}{(2\times 2)!} f''''[z_0] \text{ (* for n=2 *)}$$

$$\frac{i\pi}{12}$$

$$\frac{2\pi i}{(2\times 3)!} f''''''[z_0] \text{ (* for n=3 *)}$$

$$-\frac{i\pi}{360}$$

$$\frac{2\pi i}{(2\times 4)!} f''''''''[z_0] \text{ (* for n=4 *)}$$

$$\frac{i\pi}{20160}$$

$$\texttt{dirk[n\_, N\_, z\_] = Sequence}\left[\frac{2\pi i}{(2n)!}\texttt{D[f[z], \{z, 2 n\}], \{n, N, N\}}\right]$$

$$\texttt{Sequence}\left[\frac{2 i \pi \,\texttt{Cos}^{(2n)}[z]}{(2n)!}, \{n, N, N\}\right]$$

```
dirk[0, 0, 0]
```

Sequence[2 ⅈ π, {0, 0, 0}]

```
dirk[3, 3, 0]
```

$$\texttt{Sequence}\left[-\frac{i\pi}{360}, \{3, 3, 3\}\right]$$

```
dirk[4, 4, 0]
```

$$\texttt{Sequence}\left[\frac{i\pi}{20160}, \{4, 4, 4\}\right]$$

The green cell above expresses the text answer, with the proviso that it is evaluated at z=0. The ‘N’ factors are only present for stability of form.

8 - 19 Integration. Different contours
Integrate.

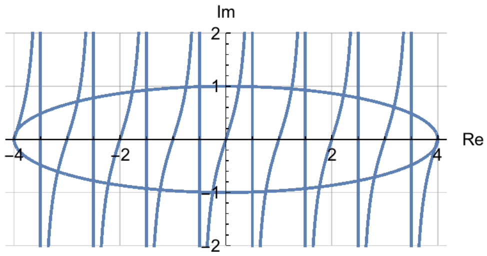
9. $\oint_C \frac{\texttt{Tan}[\pi z]}{z^2}\, dz$, C the ellipse $16 x^2 + y^2 == 1$ clockwise

```
Clear["Global`*"]
```

$$\texttt{p1 = Plot}\left[\texttt{y /. Solve}\left[\left(\frac{x}{4}\right)^2 + (y)^2 == 1\right], \{x, -4, 4\},\right.$$
```
    AspectRatio → Automatic, ImageSize → 250, AxesLabel → {"Re", "Im"},
    GridLines → Automatic, PlotRange → {-2, 2}];
```

```
p2 = Plot[Tan[π z], {z, -4, 4}];
```

```
Show[p1, p2]
```



By looking at the plot, the tangent’s asymptotes, which will be the location of its nonanalyticities, are all outside the elliptic domain of interest. There does remain the origin, however, as the point of difficulty. Recasting numbered line (1’) of p. 646,

$2\pi i\,(f'[z_0]) == \oint_C \frac{f[z]}{(z-z_0)^2}\,dz$

gives me the working target. Again z_0 will equal 0.

```
z0 = 0
0

f[z_] = Tan[π z]
Tan[π z]

f'[z]
π Sec[π z]^2

f'[z0]
π

2 π i (f'[z0])
2 i π^2
```

The s.m. tips me off to a gotcha. Boxed theorem 1, p. 664, which includes numbered line (**1**'), applies to curves integrated in the counterclockwise direction along their paths. The current problem specifies a clockwise direction of travel, meaning that I have to negate the result to get the correct answer.

```
-2 π i (f'[z0])
```

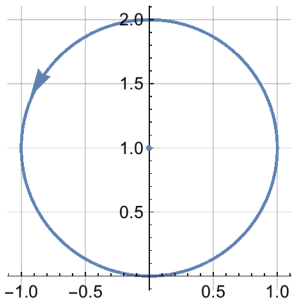
$-2\,i\,\pi^2$

11. $\oint_C \frac{(1+z)\,\mathrm{Sin}[z]}{(2z-1)^2}\,dz$, C : $|z - i| == 2$ counterclockwise

```
Clear["Global`*"]
```

Again reminding myself of the circle formula: **ParametricPlot[{r Cos[t] + a, r Sin[t] + b}, {t, 0, 2 π}**

```
ParametricPlot[{1 Cos[z], 1 Sin[z] + 1},
 {z, 0, 2 π}, ImageSize → 150, GridLines → Automatic,
 Epilog → {{RGBColor[95/255, 130/255, 179/255], Arrowheads[0.1],
    Arrow[{{-0.87, 1.5}, {-0.93, 1.4}}]},
   {RGBColor[95/255, 130/255, 179/255], Point[{0, 1}]}}]
```



I believe that $2\pi i\,(f'[z_0]) == \oint_C \frac{f[z]}{(z-z_0)^2}\,dz$

can be retained as the working target. This time $z_0 = \frac{1}{2}$. The sine function is entire, and the polynomial (1+z) is nonthreatening, so I will assume analyticity for f[z].

f[z_] = (1 + z) Sin[z]

(1 + z) Sin[z]

$z_0 = \frac{1}{2}$

$\frac{1}{2}$

f'[z]

(1 + z) Cos[z] + Sin[z]

f'[z_0]

$\frac{3}{2}\,\mathrm{Cos}\left[\frac{1}{2}\right] + \mathrm{Sin}\left[\frac{1}{2}\right]$

$2\pi i\,(f'[z_0])$

$2\,i\,\pi\left(\frac{3}{2}\,\mathrm{Cos}\left[\frac{1}{2}\right] + \mathrm{Sin}\left[\frac{1}{2}\right]\right)$

Simplify[%]

$i\,\pi\left(3\,\mathrm{Cos}\left[\frac{1}{2}\right] + 2\,\mathrm{Sin}\left[\frac{1}{2}\right]\right)$

```
N[%]
```

```
0. + 11.2833 i
```

A strange outcome. Though the symbolic result matches the answer in the text, the numerical result does not. Seems too simple to misread it. WolframAlpha gives the same answer as shown in yellow.

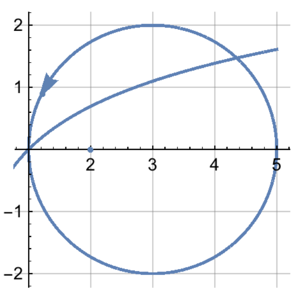
13. $\oint_C \frac{\text{Log}[z]}{(z-2)^2} dz$, C : | z − 3 | == 2 counterclockwise

```
Clear["Global`*"]
```

```
p1 =
  ParametricPlot[{2 Cos[z] + 3, 2 Sin[z] + 0}, {z, 0, 2 π}, ImageSize → 150,
   GridLines → Automatic, Epilog → {{RGBColor[95/255, 130/255, 179/255],
      Arrowheads[0.1], Arrow[{{1.213, 0.9}, {1.15, 0.8}}]},
     {RGBColor[95/255, 130/255, 179/255], Point[{2, 0}]}}];
```

```
p2 = Plot[Log[z], {z, 0, 5}];
```

```
Show[p1, p2]
```



The log function is analytic on any open set of its domain.

I believe that $2 \pi i (f'[z_0]) == \oint_C \frac{f[z]}{(z-z_0)^2} dz$

can be retained as the working target. This time $z_0=2$.

```
f[z_] = Log[z]
```

```
Log[z]
```

```
z0 = 2
```

```
2
```

```
f'[z]
```

$\frac{1}{z}$

```
f'[z0]
```

$\frac{1}{2}$

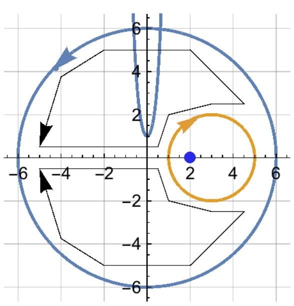
```
2 π i (f'[z0])
```

$i\pi$

15. $\oint_C \frac{\text{Cosh}[4z]}{(z-4)^3}\,dz$,

C consists of $|z| == 6$ counterclockwise and $|z-3| = 2$ clockwise

```
Clear["Global`*"]

p1 =
  ParametricPlot[{{6 Cos[z] + 0, 6 Sin[z] + 0}, {2 Cos[z] + 3, 2 Sin[z] + 0}},
   {z, 0, 2 π}, ImageSize → 150, GridLines → Automatic,
   Epilog → {{RGBColor[95/255, 130/255, 179/255], Arrowheads[0.1],
      Arrow[{{-4.25, 4.2}, {-4.47, 4.0}}]}, {RGBColor[40/255, 40/255, 230/255],
      PointSize[0.04], Point[{2, 0}]}, {RGBColor[223/255, 155/255, 52/255],
      Arrowheads[0.085], Arrow[{{2.15, 1.75}, {2.48, 1.95}}]}}];

p2 = Plot[Cosh[4 z], {z, -6, 6}];

p3 = Graphics[Line[{{2, 5}, {-2, 5}, {-4, 3.75}, {-5, 0.5},
     {0.5, 0.5}, {1, 2}, {3, 2.5}, {4.5, 2.5}, {2, 5}}]];

p4 = Graphics[Line[{{-5, -0.5}, {0.5, -0.5}, {1, -2},
     {3, -2.5}, {4.5, -2.5}, {2, -5}, {-2, -5}, {-4, -3.75}}]];

p5 = Graphics[{{Arrowheads[0.085], Arrow[{{-4, -3.75}, {-5, -0.5}}]},
    {Arrowheads[0.085], Arrow[{{-4, 3.75}, {-5, 0.5}}]}}];

Show[p1, p2, p3, p4, p5]
```



According to Wikipedia, the (complex) hyperbolic cosine is holomorphic, which I believe means it is also analytic.

I will leave $2\pi i\,(f'[z_0]) == \oint_C \frac{f[z]}{(z-z_0)^2}\,dz$ as the working target. The z_0 I will set as 2, and pull a squared term out as part of the function f[z].

$$f[z\_] = \frac{\text{Cosh}[4\,z]}{(z-4)^2}$$

$$\frac{\text{Cosh}[4\,z]}{(-4+z)^2}$$

$$z_0 = 2$$

$$2$$

$$f'[z]$$

$$-\frac{2\,\text{Cosh}[4\,z]}{(-4+z)^3} + \frac{4\,\text{Sinh}[4\,z]}{(-4+z)^2}$$

$$f'[z_0]$$

$$\frac{\text{Cosh}[8]}{4} + \text{Sinh}[8]$$

$$2\pi i\,(f'[z_0])$$

$$2\,i\,\pi\left(\frac{\text{Cosh}[8]}{4} + \text{Sinh}[8]\right)$$

I would propose the yellow cell as the answer if only the teal and not the orange domain was in the problem. But as explained on *http://www.math.unm.edu/~nitsche/courses/313/s16/lec19_int5.pdf*, the orange domain makes a difference. Essentially, working the problem can be done using the Cauchy-Goursat theorem, which involves branch cuts. One path around the branch cuts is counterclockwise and the other is clockwise, and a difference in sign makes them cancel out, giving a total of 0, which is what the text reports. Rough paths are shown in the sketch.

17. $\oint_C \frac{e^{-z}\,\text{Sin}[z]}{(z-4)^3}\,dz$,

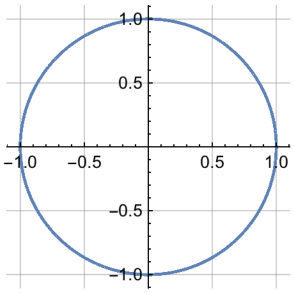
C consists of $|z| == 5$ counterclockwise and $|z-3| == \frac{3}{2}$ clockwise

This one is somewhat similar to the last. The text answer for this problem quotes numbered line (6) in section 14.2. This deals with Cauchy's integral theorem for doubly connected domains. However, in that section it was specifically stated that both paths had the same orientation. In this problem the orientations of inner and outer path are opposite. Again, *http://www.math.unm.edu/~nitsche/courses/313/s16/lec19_int5.pdf* does give some support for zero total integral value, again, making use of Cauchy-Goursat. As for analyticity, Wikipedia vouches for the exponential function and the trig functions.

19. $\oint_C \frac{e^{3z}}{(4z - \pi i)^3} dz$, C : $|z| == 1$ counterclockwise

```
Clear["Global`*"]
```

```
p1 = ParametricPlot[{1 Cos[z] + 0, 1 Sin[z] + 0},
  {z, 0, 2 π}, ImageSize → 150, GridLines → Automatic]
```



I believe that $\frac{2 \pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f[z]}{(z - z_0)^{n+1}} dz$

can be retained as the working target. This time the function will be a little complicated. I need to calculate what n is equal to.

I need to try to unscramble the denominator,

$(4z - \pi i)^3 == (4z - \pi i)(4z - \pi i) ==$
$4\left(z - \frac{\pi i}{4}\right) 4\left(z - \frac{\pi i}{4}\right) 4\left(z - \frac{\pi i}{4}\right) == 4^3 \left(z - \frac{\pi i}{4}\right)^3$

I can pull the 4^3 out of the denominator and modify the coefficient accordingly

$\text{coef} = \frac{2 \pi i}{4^3 n!}$

$\frac{i \pi}{32 n!}$

Looking at what I have left, n=2 and $z_0 = \frac{\pi i}{4}$,

$z_0 = \frac{\pi i}{4}$

$\frac{i \pi}{4}$

$\text{coef} = \text{coef} /. n \to 2$

$\frac{i \pi}{64}$

And inside the integral, the numerator is the function,

```
f[z_] = e^(3 z)
```

e^{3z}

```
f''[z]
```

$9e^{3z}$

Since n=2, I want the second derivative of z_0,

```
f''[z0]
```

$9e^{3z_0}$

```
coef f''[z0]
```

$$\frac{9}{64}\, i\, e^{\frac{3 i \pi}{4}}\, \pi$$

The green cell above matches the text answer, but the text then recasts it.

$$\text{ans} = \frac{-9\pi(1+i)}{64\sqrt{2}}$$

$$-\frac{\left(\frac{9}{64}+\frac{9 i}{64}\right)\pi}{\sqrt{2}}$$

```
FullSimplify[%]
```

$$-\frac{9}{64}(-1)^{1/4}\pi$$

The answers agree if

$$-(-1)^{1/4} == i\, e^{\frac{3 i \pi}{4}}$$

True

$$\text{FullSimplify}\left[i\, e^{\frac{3 i \pi}{4}}\right]$$

$$-(-1)^{1/4}$$